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REDUCIBILITY OF MATRIX EQUATIONS CONTAINING SEVERAL PARAMETERS. (U)

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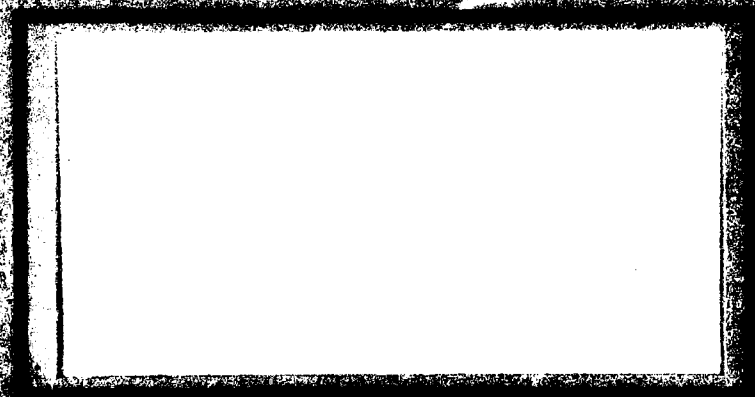
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REDUCIBILITY OF MATRIX EQUATIONS

CONTAINING SEVERAL PARAMETERS

Thesis

AFIT/GE/MA/81D-1

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Lieutenant USAF

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REDUCIBILITY OF MATRIX EQUATIONS  
CONTAINING SEVERAL PARAMETERS

THESIS

Presented to the Faculty of the School of Engineering  
of the Air Force Institute of Technology  
Air University  
in Partial Fulfillment of the  
Requirements for the Degree of  
Master of Science

by

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December 1981

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## Preface

Mathematics has and will always be a fascinating subject for me. The almost non-existence of methods to efficiently reduce matrices, especially large matrices in several parameters astounded me, since a 70-dimensional, multi-parameter state matrix is commonplace. My main motivation is to provide a tool with which these matrices can be reduced to a more sparse, manageable form, while maintaining system characteristics, reducing CP time and memory, thereby reducing cost. This thesis primarily extends the work of Jones, Leuthauser, Frost and Storey, and Dolan.

This work was sponsored by the Air Force Medical Research Laboratory (AMRL), Wright-Patterson AFB, Ohio. I would like to thank Dr. Dan Repperger of the AMRL for his unfailing computer support and expertise, and Professor Albert Moore and Major Kenneth Castor for taking time to review this report. Most of all I would like to thank God for his never-ending mental, spiritual, and emotional support during my stay at Wright-Patterson AFB.

Dedication of this work goes to my thesis advisor, Dr. John Jones, Jr., whose wealth of knowledge, enthusiasm and moral support was indispensable to successful completion of this project. Finally, I thank my typist, Jane Manemann, for her gargantuan effort in bringing this work to fruition.

Charles A. Lew

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### Abstract

Two methods of reducing state matrix equations containing several parameters are presented in this thesis. Pursuant to the first method, the solutions, obtained by iterative and non-iterative methods, of the algebraic Riccati matrix equation and Lyapunov matrix equation are relied upon heavily. The second method hinges on an equivalence transformation to a Smith form, coupled with the notion of zeros of a matrix. Main results focus on the two parameter case but can be extended to any number of parameters.

REDUCIBILITY OF MATRIX EQUATIONS CONTAINING  
SEVERAL PARAMETERS

I. Introduction

Reducibility of the equation

$$\dot{x} = Ax \quad (1.0)$$

where  $A$  is a constant matrix has been widely studied. Classical methods for solving Eq (1.0) involved the use of similarity transforms using eigenvectors and eigenvalues. One wishes to reduce  $A$  to a sparser form, maybe block diagonal or even diagonal form to ease computation time, memory requirements, and cost of solving Eq (1.0). Much study has been done for the case where matrix contains polynomial elements of a single parameter, see Jones (Ref 13) or Browne (Ref 8:138-149) for example. Presently, there is no known method for reducing matrix equations that have polynomial elements in many parameters, say  $z_1, z_2, \dots, z_q$ , or  $\underline{z}$  for short. It is the purpose of this thesis to provide a method for reducing matrix equations in several parameters. The main emphasis will deal with the two-parameter case, and the extension to the multi-parameter case can be done by induction using the basic results found here.

Throughout this thesis it will be assumed that matrices can be rectangular and belong to the ring  $R[\underline{z}]$ . Also, conditions of stability and nonsingularity in coefficient matrices will not be imposed.

Two methods of reduction will be presented: (1) the Riccati-Lyapunov Method and (2) the Smith Form Method. Iterative and non-iterative solutions of the Riccati and Lyapunov equations pertaining to the first method will be dealt with in Chapter II. Chapter III shows

how to reduce a system using method (1) and method (2), and gives an application to singular systems. Finally, Chapter IV summarizes what has been done and gives recommendations for future study.

## II. Solutions of Riccati and Lyapunov Equations

The purpose of this chapter is to present new and existing methods for obtaining solutions to the algebraic Riccati matrix equation and the Lyapunov matrix equation. Section 1 will deal with the Riccati equation, presenting four methods of solution. The first method will use generalized inverses in its solution. The second method will implement a function of a matrix polynomial to obtain its solution. The third method draws upon a power series expansion to arrive at its solution. Finally, the last method utilizes an iterative technique to zero in on a solution. Each method will be followed by an example. Section 2 deals with solutions of the Lyapunov matrix equation. Since the form of the Lyapunov equation is closely tied to the Riccati equation, the methods developed in Section 1 will also apply for solving the Lyapunov equation. An additional method for solving the Lyapunov equation that will be presented is the Idempotent-Nilpotent method. Examples will follow, including a case involving one parameter.

### Section 1. Solutions of Riccati Matrix Equations

The Riccati matrix equation occurs most commonly in modern control theory, in the design of optimal controllers (Ref 24:18) and estimators (Ref 29) for linear time-invariant systems with quadratic cost. Other uses of the Riccati equation arise from singular perturbation methods (Ref 25) and filtering theory (Ref 20). New uses of the Riccati equation crop up in particle beam transport processes (Ref 6:28). The form of the steady state of algebraic Riccati matrix equation we will use is basically the following:

$$AX + XB + C + XDX = 0 \quad (2.0)$$

Other equations similar in structure to Eq (2.0) will follow. The form of Eq (2.0) actually comes from

$$A^T K + KA - KBR^{-1}B^T K + Q = 0 \quad (2.1)$$

which is a result of a minimization of a quadratic cost function that is related to the linear constant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \quad (2.2)$$

Generalized Inverse Method. The emergence of generalized inverses came about to fill the void in the solution methodology of systems with singular or rectangular matrix coefficients. Applications of generalized inverses include iterative methods for solving nonlinear equations, interval linear programming, integral solution of linear equations (Ref 7:27,90,93,96), estimation theory (Ref 10:1) and systems science (Ref 21).

For every finite matrix A (not necessarily square) whose elements contain coefficients belonging to the set of complex numbers, there exists a unique matrix X satisfying the four Penrose equations (Ref 7:7):

$$AXA = A \quad (2.3)$$

$$XAX = X \quad (2.4)$$

$$(AX)^* = AX \quad (2.5)$$

$$(XA)^* = XA \quad (2.6)$$

where  $A^*$  denotes the conjugate transpose of A. The matrix  $X = A^-$

satisfying Eq (2.3) is called a generalized inverse or a 1-inverse, not necessarily unique. The matrix  $X = A_r$ , not necessarily unique, satisfying Eqs (2.3), (2.4) is called a reflexive generalized inverse or a 1,2-inverse. The matrix  $X = A_w$ , not necessarily unique, satisfying Eqs (2.3), (2.4), (2.5) is called a left-weak generalized inverse of a 1,2,3-inverse. The matrix  $x = A_n$ , not necessarily unique, satisfying Eqs (2.3), (2.4), (2.6) is called a right weak generalized inverse, or a 1,2,4-inverse. Finally, the matrix  $X = A^+$  satisfying Eqs (2.3) through (2.6) is called a Moore-Penrose inverse. The matrix  $A^+$  is unique, and is the same as  $A^{-1}$ , the ordinary inverse (Ref 7:7).

#### Theorem 2.1

$X$  is a solution of the equation

$$AX + XB + C - XDX = 0 \quad (2.7)$$

iff

$$(A-I)K(LDK)_r L + K(LDK)_r LB + C = 0 \quad (2.8)$$

where

$$X = K(LDK)_r L \quad (2.9)$$

and  $K$  and  $L$  are arbitrary matrices of appropriate dimension. Equation (2.8) is called a consistency condition for Eq (2.7).

#### Proof (Necessity)

Substituting Eq (2.9) into Eq (2.7) we have

$$AK(LDK)_r L + K(LDK)_r LB + C - K(LDK)_r LDK(LDK)_r L = 0 \quad (2.10)$$

Using Eq (2.5) we have

$$AK(LDK)_r L + K(LDK)_r LB + C - K(LDK)_r L = 0 \quad (2.11)$$

$$(A-I)K(LDK)_r L + K(LDK)_r LB + C = 0 \quad (2.12)$$

Proof (Sufficiency)

Expanding Eq (2.8) we have

$$AK(LDK)L - K(LDK)L + K(LDK)LB + C = 0 \quad (2.13)$$

$$AK(LDK)_r L + K(LDK)_r LB + C - K(LDK)_r LDK(LDK)_r L = 0 \quad (2.14)$$

Comparing Eq (2.14) with Eq (2.7) we have  $X = K(LDK)_r L$ .

Method for Obtaining  $A_r^-$ . One of the best existence theorems of a generalized inverse is given by Jones (Ref 14). The following treatment draws upon the work done by Jones. Getting the reflexive inverse is easy once you have generated two distinct generalized inverses or 1-inverses. Let  $A_1^-$  and  $A_2^-$  be two 1-inverses to find. We can find two non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (2.15)$$

where r is the rank of A, then  $A_1^-$  is given by

$$A_1^- = Q \begin{bmatrix} I_r & U_1 \\ V_1 & W_1 \end{bmatrix} P \quad (2.16)$$

for arbitrary  $U_1, V_1, W_1$  of appropriate dimension. Similarly

$$A_2^- = Q \begin{bmatrix} I_r & U_2 \\ V_2 & W_2 \end{bmatrix} P \quad (2.17)$$

for arbitrary  $U_2, V_2, W_2$ . Now that we have two 1-inverses we can generate  $A_r$  by the following equation:

$$A_r = A_1^- A A_2^- \quad (2.18)$$

See Sontag (Ref 27) for obtaining the generalized inverse of matrices containing parameters.

#### Example 2.1

Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (2.19)$$

Then a solution to  $AX + XB + C - XDX = 0$  is given by  $X = K(LDK)_r L$  iff Eq (2.6) holds. Choosing

$$K = I \quad L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (2.20)$$

we have

$$(A-I)K(LDK)_r L + K(LDK)_r LB + C = 0 \quad (2.21)$$

Hence

$$X = K(LDK)_r L = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (2.22)$$

As an extension to Eq (2.5) consider the generalized Riccati matrix equation of order three in the following theorem:

Theorem 2.2

X is a solution of the equation

$$AX + XB + C - XDX - XEXFX = 0 \quad (2.23)$$

iff

$$(A - I - K(LEK)_r LF)K(LEK)_r L + K(LEK)_r L(B + DK(LEK)_r L) + C = 0 \quad (2.24)$$

where

$$X = K(LEK)_r L \quad (2.25)$$

Proof follows by similar methods in the proof of Theorem 2.1.

This method of solution can be extended to fourth and higher order equations. As the order of the generalized Riccati matrix equations increases, so increases the complexity of the consistency condition, placing tighter constraints on X. This implies it will be less easy to find matrices K and L to fit the consistency conditions corresponding to their appropriate equations. As the notion is rather intuitive, no proof will be given.

Example 2.2

$$\begin{aligned} \text{Let } A &= \begin{bmatrix} 3/2 & 1 \\ 1/2 & 2 \end{bmatrix} & B &= \begin{bmatrix} -11/4 & -11/4 \\ -5/2 & -5/2 \end{bmatrix} & C &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ D &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} & E &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & F &= \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \end{aligned} \quad (2.26)$$

Then a solution to  $AX + XB + C - XDX - XEXFX = 0$  is given by

$X = K(LEK)_r L$  iff Eq (2.20) holds. Choosing

$$K = I \quad L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (2.27)$$

we have

$$(A-I-K(LEK)_r LF)K(LEK)_r L + K(LEK)_r L(B+DK(LEK)_r L) + C = 0 \quad (2.28)$$

Hence

$$X = K(LEK)_r L = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (2.29)$$

f(R) Method. This method relies on the formation of a state matrix similar in form to the one in the Hamiltonian system of equations. Referring back to the linear system in Eq (2.2), one can form the control Hamiltonian (Ref 2:238) and obtain the Hamiltonian system of equations:

$$\begin{matrix} x \\ y \end{matrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = H \begin{bmatrix} X \\ Y \end{bmatrix} \quad (2.30)$$

The similar system we will use is the following:

$$\dot{\Phi} = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \Phi \quad (2.31)$$

Jones (Ref 13) has an excellent paper on the necessary and sufficient conditions concerning solutions to Eq (2.0). Dolan (Ref 11:65)

has extended Jones' treatment of the cases involving square matrices to the rectangular case.

Theorem 2.3 (Jones (Ref 13))

Let  $f_Y(\lambda)$  be any polynomial of degree  $n \geq 1$  in  $\lambda$  with coefficients belonging to the field of complex numbers such that for square matrices  $A, B, C, D$ ,

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix}, \quad f_Y(R) = \begin{bmatrix} U & V \\ M & N \end{bmatrix} \quad (2.32)$$

Then a solution of

$$(X, I)f_Y(R) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.33)$$

with  $U^{-1}$  or  $M^{-1}$  existing, or a solution of

$$f_Y(R) \begin{bmatrix} I \\ -X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.34)$$

with  $M^{-1}$  or  $N^{-1}$  existing is also a solution of Eq (2.0).

Experience in using this method has shown that Jones' theorem will not always work even if  $U^{-1}$  or  $M^{-1}$  or  $N^{-1}$  exists. An example of the inadequacy of Jones' theorem will follow after slight modification.

Theorem 2.4

Let  $f_Y(\lambda)$  be any polynomial of degree  $n \geq 1$  in  $\lambda$  with coefficients belonging to the field of complex numbers such that  $R$  and  $f(R)$  are as given in Eq (2.32). Then a common solution to Eq (2.33) with  $U^{-1}$  or  $M^{-1}$

existing and/or to Eq (2.34) with  $M^{-1}$  or  $N^{-1}$  existing is also a solution of Eq (2.0).

This method finds all solutions that Martensson's method (Ref 24) can find plus more. Illustration of the theorem is followed by an example from Martensson rewritten to conform with Eq (2.0).

Example 2.3 (Ref 24:31-32)

Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad (2.35)$$

Then

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & -3 & 0 & -1 \end{bmatrix} \quad (2.36)$$

and

$$\det(R - \lambda I) = (\lambda^2 - 4)^2 \quad (2.37)$$

The polynomials  $f_\gamma(\lambda)$  are found by constructing a multiplication table consisting of the distinct factors of the characteristic polynomial, i.e.,

Table I

Multiplication Factor Table of Eigenvalues of R

X	$\lambda-2$	$\lambda+2$
$\lambda-2$	$(\lambda-2)^2$	$\lambda^2-4$
$\lambda+2$	$\lambda^2-4$	$(\lambda+2)^2$

Examining the upper or lower triangular portion of the table we find that there are three distinct matrix polynomial cases to consider:

$$\begin{aligned}
 \text{Case 1} \quad f_Y(R) &= R^2 - 4I \\
 \text{Case 2} \quad f_Y(R) &= R^2 - 4R + I \\
 \text{Case 3} \quad f_Y(R) &= R^2 + 4R + 4I
 \end{aligned} \tag{2.38}$$

Case 1.  $R^2 - 4I$

$$\begin{bmatrix} 4 & 3 & 0 & 3 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \tag{2.39}$$

$$X = -NM^{-1} = - \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1/3 \\ 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.40}$$

$$X = M^{-1}U = \begin{bmatrix} 0 & -1/3 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.41}$$

Case 2.  $R^2 - 4R + 4I$

$$\begin{bmatrix} 16 & 3 & 4 & 7 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad (2.42)$$

$$X = -VU^{-1} = -\begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix} \frac{1}{64} \begin{bmatrix} 4 & -3 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad (2.43)$$

$$X = -NM^{-1} = -\begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 4 & -7 \\ -1 & 4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 0 & 0 \\ -24 & 79 \end{bmatrix} \quad (2.44)$$

$$X = M^{-1}U = \frac{1}{9} \begin{bmatrix} 4 & -7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 16 & 3 \\ 0 & 4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 64 & -16 \\ -16 & 13 \end{bmatrix} \quad (2.45)$$

Case 3.  $R^2 + 4R + 4I$

$$\begin{bmatrix} 0 & 3 & -4 & -1 \\ 0 & 12 & -7 & -4 \\ 0 & 0 & 16 & 0 \\ 0 & -12 & 3 & 4 \end{bmatrix} = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad (2.46)$$

$$X = -NM^{-1} = -\begin{bmatrix} 16 & 0 \\ 3 & 4 \end{bmatrix} \frac{1}{9} \begin{bmatrix} -4 & 1 \\ 7 & -4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 64 & -16 \\ -16 & 13 \end{bmatrix} \quad (2.47)$$

$$X = M^{-1}U = \frac{1}{9} \begin{bmatrix} -4 & 1 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad (2.48)$$

Examining the common solutions to the above three cases we find three solutions satisfy Eq (2.0):

$$X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad X_3 = 1/9 \begin{bmatrix} 64 & -16 \\ -16 & 13 \end{bmatrix} \quad (2.49)$$

Note that because of the noncommonality of  $X$  in Eq (2.44), it is not a solution to Eq (2.0) as Jones claims, even though  $M^{-1}$  exists. Also let it be known that Martensson's method only gave solution  $X_1$ .

The next theorem is a slight modification of a theorem by Dolan (Ref 11:65), who treats the rectangular case. An example follows.

#### Theorem 2.5

Let  $f_Y(\lambda)$  be any polynomial of degree  $n \geq 1$  in  $\lambda$  such that  $R$  and  $f_Y(R)$  are as given in Eq (2.32) for  $A_{m \times m}$ ,  $B_{n \times n}$ ,  $C_{m \times n}$ ,  $D_{n \times m}$ . Then a common solution to Eq (2.33) with  $U^{-1}$  existing and/or to Eq (2.34) with  $N^{-1}$  existing for  $X_{m \times n}$  is also a solution to Eq (2.0).

#### Example 2.4

Let

$$A = [-1] \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad C = [0 \ -1] \quad D = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \quad (2.50)$$

Then

$$R = \begin{bmatrix} -3 & -1 & -2 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad (2.51)$$

and

$$\det(R - \lambda I) = (\lambda + 3)(\lambda + 2)(\lambda + 1) \quad (2.52)$$

Using Theorem 2.5, two solutions to Eq (2.0) are

$$X_1 = [0 \ 1], X_2 = [1 \ 0] \quad (2.53)$$

Beavers and Denman (Ref 4:340), using a similar partitioning scheme to our matrix R, claim that if there are  $2n$  distinct eigenvalues of R then there are  ${}_n C_n = (2n)!/(n!n!)$  possible solutions to an  $n \times n$  matrix Riccati equation. Then they show an example (Ref 3:141-143) for  $n = 2$ , constructing six solutions. Let it be known that only two of the six claimed solutions actually satisfy the matrix Riccati equation, and that those two solutions were readily found by the  $f(R)$  method. A better proposal to the number of possible solutions is the following: For any matrix R, if there are  $p$  distinct eigenvalues of R, then the number of possible solutions is given by  $p(p+1)/2$ . It is easy to derive this number by examining the Multiplication Factor Table of the eigenvalues of R.

Power Series Method. This method relies on a power series solution to solve Eq (2.54) and Eq (2.55). As the quadratic has been studied by Dolan (Ref 11) and the cubic case by Leuthauser (Ref 18), the extension to the fourth order case will be presented. Note that the higher order case can be reduced to the Riccati or Lyapunov equation by setting the coefficient matrices of those terms to zero. Consequently, this method can be extended to cases involving fifth order terms to finally  $n$ th order terms. The Eqs (2.54) and (2.55) arise from the transport equation of particles involving three scattering processes (Ref 28:633). What follows is an extension of Leuthauser (Ref 18:59-64).

#### Theorem 2.6

Consider the equations

$$\hat{A}X - XB = C + \hat{X}F\hat{X} + \hat{X}D\hat{X}E\hat{X} + \hat{X}L\hat{X}G\hat{X}H\hat{X} \quad (2.54)$$

$$\hat{A}\hat{Y} - \hat{Y}\hat{B} = C + \hat{Y}\hat{F}\hat{Y} + \hat{Y}\hat{D}\hat{Y}\hat{E}\hat{Y} + \hat{Y}\hat{L}\hat{Y}\hat{G}\hat{Y}\hat{H}\hat{Y} \quad (2.55)$$

Equation (2.54) has a solution  $\hat{X}$ , and Eq (2.55) has a solution  $\hat{Y}$  if the equation

$$(A-\lambda I)X(\lambda) - Y(\lambda)(B-\lambda I) = C + \hat{X}\hat{F}\hat{X} + \hat{X}\hat{D}\hat{X}\hat{E}\hat{X} + \hat{X}\hat{L}\hat{X}\hat{G}\hat{X}\hat{H}\hat{X} \quad (2.56)$$

have solutions  $X(\lambda)$  and  $Y(\lambda)$  where

$$X(\lambda) = X_0 + \lambda X_1 + \lambda^2 X_2 + \dots + \lambda^p X_p \quad (2.57)$$

$$Y(\lambda) = Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \dots + \lambda^p Y_p \quad (2.58)$$

Then  $\hat{X}$  and  $\hat{Y}$  are given by

$$\hat{X} = X_0 + X_1 B + X_2 B^2 + \dots + X_p B^p \quad (2.59)$$

$$\hat{Y} = Y_0 + A Y_1 + A^2 Y_2 + \dots + A^p Y_p \quad (2.60)$$

To obtain the  $X_i$  and  $Y_i$ , we must examine Eq (2.56). Equating coefficients of like powers of  $\lambda$  we have the following set of equations:

$$\begin{array}{rcll} AX_0 & - & Y_0 B & = C + \hat{X}\hat{F}\hat{X} + \hat{X}\hat{D}\hat{X}\hat{E}\hat{X} + \hat{X}\hat{L}\hat{X}\hat{G}\hat{X}\hat{H}\hat{X} \\ AX_1 & - & X_0 & - Y_1 B + Y_0 = 0 \\ AX_2 & - & X_1 & - Y_2 B + Y_1 = 0 \\ \vdots & & \vdots & \\ \vdots & & \vdots & \\ AX_{p-1} & - & X_{p-2} & - Y_{p-1} B + Y_{p-2} = 0 \\ AX_p & - & X_{p-1} & - Y_p B + Y_{p-1} = 0 \\ & & - X_p & + Y_p = 0 \end{array} \quad (2.61)$$

Multiplying the equations in Eq (2.61) on the right by  $I, B, B^2, \dots, B^{p+1}$  gives

$$\begin{array}{rclcl}
AX_0 & & - Y_0 B & & = C + \hat{X}\hat{F}\hat{X} + \hat{X}\hat{D}\hat{X}\hat{E}\hat{X} + \hat{X}\hat{L}\hat{X}\hat{G}\hat{X}\hat{H}\hat{X} \\
AX_1 B & - X_0 B & - Y_1 B^2 & + Y_0 B & = 0 \\
AX_2 B^2 & - X_1 B^2 & - Y_2 B^3 & + Y_1 B^2 & = 0 \\
\vdots & \vdots & \vdots & \vdots & \\
AX_{p-1} B^{p-1} & - X_{p-2} B^{p-1} & - Y_{p-1} B^p & + Y_{p-2} B^{p-1} & = 0 \\
AX_p B^p & - X_{p-1} B^p & - Y_p B^{p+1} & + Y_{p-1} B^p & = 0 \\
& & - X_p B^{p+1} & + Y_p B^{p+1} & = 0
\end{array} \tag{2.62}$$

Columnwise addition gives

$$\begin{aligned}
& [AX_0 + AX_1 B + AX_2 B^2 + \dots + AX_{p-1} B^{p-1} + AX_p B^p] \\
& - [X_0 B + X_1 B^2 + \dots + X_{p-2} B^{p-1} + X_{p-1} B^p + X_p B^{p+1}] \\
& = C + \hat{X}\hat{F}\hat{X} + \hat{X}\hat{D}\hat{X}\hat{E}\hat{X} + \hat{X}\hat{L}\hat{X}\hat{G}\hat{X}\hat{H}\hat{X}
\end{aligned} \tag{2.63}$$

Factoring, we have

$$\begin{aligned}
& A(X_0 + X_1 B + X_2 B^2 + \dots + X_{p-1} B^{p-1} + X_p B^p) \\
& - (X_0 + X_1 B + \dots + X_{p-2} B^{p-2} + X_{p-1} B^{p-1} + X_p B^p) B \\
& = C + \hat{X}\hat{F}\hat{X} + \hat{X}\hat{D}\hat{X}\hat{E}\hat{X} + \hat{X}\hat{L}\hat{X}\hat{G}\hat{X}\hat{H}\hat{X}
\end{aligned} \tag{2.64}$$

Thus we see that  $\hat{X}$  is a solution to Eq (2.54). Similarly we can derive  $\hat{Y}$  as a solution by multiplying the equations in Eq (2.61) on the left by  $I, A, A^2, \dots, A^{p+1}$  and adding columnwise. This gives

$$\begin{aligned}
& A[Y_0 + AY_1 + A^2 Y_2 + \dots + A^{p-1} Y_{p-1} + A^p Y_{p+1}] \\
& - [Y_0 + AY_1 + A^2 Y_2 + \dots + A^{p-1} Y_{p-1} + A^p Y_{p+1}] B \\
& = C + \hat{Y}\hat{F}\hat{Y} + \hat{Y}\hat{D}\hat{Y}\hat{E}\hat{Y} + \hat{Y}\hat{L}\hat{Y}\hat{G}\hat{Y}\hat{H}\hat{Y}
\end{aligned} \tag{2.65}$$

insuring that  $\hat{Y}$  is a solution to Eq (2.55).

### Example 2.5

Let

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} & B &= \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} & C &= \begin{bmatrix} 36 & -66 \\ -22 & -3 \end{bmatrix} \\ D &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & E &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & F &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L = G = H \end{aligned} \quad (2.66)$$

in the equations

$$\hat{A}\hat{X} - \hat{X}B = C + \hat{X}F\hat{X} + \hat{X}D\hat{X}E + \hat{X}L\hat{X}G\hat{X}H \quad (2.67)$$

$$\hat{A}\hat{Y} - \hat{Y}B = C + \hat{Y}F\hat{Y} + \hat{Y}D\hat{Y}E + \hat{Y}L\hat{Y}G\hat{Y}H \quad (2.68)$$

Then for  $p = 3$

$$\begin{aligned} X(\lambda) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda^3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ Y(\lambda) &= \begin{bmatrix} -5 & -11 \\ -3 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 4 & 4 \\ 1 & -1 \end{bmatrix} + \lambda^2 \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} + \lambda^3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (2.70)$$

Hence

$$\hat{X} = X_0 + X_1 B + X_2 B^2 + X_3 B^3 = \begin{bmatrix} -2 & -8 \\ 0 & 5 \end{bmatrix} \quad (2.71)$$

$$\hat{Y} = Y_0 + AY_1 + A^2 Y_2 + A^3 Y_3 = \begin{bmatrix} -2 & -8 \\ 0 & 5 \end{bmatrix} \quad (2.72)$$

Note that we could choose as many or as few terms we wish by adjusting  $p$ .

Iterative Method. This method for solving the Riccati was originally investigated by Leuthauser (Ref 18:56-58) and Dolan (Ref 11:37-41+) using ideas from Lancaster (Ref 17). A program was written in Fortran using double precision arithmetic and implemented on the CDC 6600. Computer Processor (CP) time per iteration was very small, averaging a miniscule two thousandths of a second.

Theorem 2.7 (Leuthauser (Ref 18:50-58))

If  $f(z) = (z+a)(z-a)^{-1}$  where  $a \neq 0$  and  $a$  is real, and if

$$f(A) = (aI-A)^{-1}(aI+A) = U \quad (2.73)$$

$$f(B) = (aI+B)(aI-B)^{-1} = V \quad (2.74)$$

where  $a$  is chosen such that  $(aI-A)^{-1}$  and  $(aI-B)^{-1}$  exist, then a solution of

$$X = UXV - \frac{1}{2a} (U-I)(C+XDX)(V-I) \quad (2.75)$$

is also a solution of

$$AX + XB = C + XDX \quad (2.76)$$

We can adjust parameter " $a$ " to speed convergence. See Leuthauser (Ref 18:56-57) for proof. If conditions in Theorem 2.7 hold, then we can obtain a recursive formula for  $X$ :

$$X_{n+1} = UX_nV - \frac{1}{2a} (U-I)(C+X_nDX_n)(V-I) \quad (2.77)$$

Reinvestigating the examples from Martensson (Ref 24:31-32+), we find that we can obtain more solutions than Martensson could find and obtain a greater accuracy and faster convergence for those examples than could

Dolan (Ref 11:71). Rewriting Martensson's example (Ref 24:44) to fit Eq (2.0) we have the following:

Example 2.6

$$A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.78)$$

Martensson's eigenvalue-eigenvector method shows two non-negative definite solutions  $X_1$  and  $X_2$  where

$$X_1 = \begin{bmatrix} 3+\sqrt{2} & 1+\sqrt{2} \\ 1+\sqrt{2} & 1+\sqrt{2} \end{bmatrix} \quad X_2 = \begin{bmatrix} \sqrt{2}-1 & -\sqrt{2}+1 \\ -\sqrt{2}+1 & \sqrt{2}-1 \end{bmatrix} \quad (2.79a)$$

Dolan, using Eq (2.77) and single precision arithmetic, arrived at two solutions  $X_2$  and  $X_3$ , where

$$X_2 = \begin{bmatrix} .414 & -.414 \\ -.414 & .414 \end{bmatrix} \quad X_3 = \begin{bmatrix} -2.41 & 2.41 \\ 2.41 & -2.41 \end{bmatrix} \quad (2.79b)$$

using an initial matrix of zero for both solutions. Solution  $X_2$  was arrived at in 4 iterations using  $a = 3$ , while  $X_3$  was obtained in 8 iterations with  $a = -7$ . Although Dolan found solution  $X_3$  that Martensson could not find, he could not find solution  $X_1$ . The author's attempts at this example were very successful. Three solutions of Eq (2.76) were found. Double precision arithmetic and sixteen place accuracy were used. Results follow:

$$X_1 = \begin{bmatrix} 4.414213562373095 & 2.414213562373095 \\ 2.414213562373095 & 2.414213562373095 \end{bmatrix} \quad (2.80a)$$

using

$$x_0 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, a = .133$$

$$x_2 = \begin{bmatrix} .414213562373095 & -.414213562373095 \\ -.414213562373095 & .414213562373095 \end{bmatrix} \quad (2.80b)$$

using

$$x_0 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, a = .2977$$

$$x_3 = \begin{bmatrix} -2.414213562373095 & 2.414213562373095 \\ 2.414213562373095 & -2.414213562373095 \end{bmatrix} \quad (2.80c)$$

$$= \begin{bmatrix} -1-\sqrt{2} & 1+\sqrt{2} \\ 1+\sqrt{2} & -1-\sqrt{2} \end{bmatrix}$$

using

$$x_0 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, a = -.1$$

Number of iterations to converge were six for  $x_1$ , seven for  $x_2$  and eight for  $x_3$ . Note that this attempt produced the intersection of solutions from Martensson and Dolan. One can see that for different values of  $a$  and identical starting points we can arrive at different solutions. Also, the same solution can be arrived at using different starting points and different values of  $a$ .

Dolan (Ref 11:40) gives some rules of thumb that work quite well.

They are as follows:

- (1) Choose an  $a$  equal to the minimum of the elements of the matrices  $A$ ,  $B$ , and  $C$ .
- (2) Assure that  $U$  and  $V$  exist.
- (3) Determine if the sequence generated by Eq (2.77) converges.
- (4) If the sequence is converging, then iterate until less than a specified tolerance.
- (5) If the sequence diverges, increment  $a$  by a specified amount.

The specified amount in step 5 could be determined by the elements of the matrices  $A$ ,  $B$ , and  $C$ . For example, if the range of the elements was from .1 to 10 then an adequate adjustment would be  $\pm 1$ . But if the range of the values of the elements was from 1 to 100 then an incremental value of  $\pm 10$  was adequate.

The thrust after obtaining an  $a$  to get convergence, was to find  $a_{\min}$ , i.e., the minimum value of  $a$  to get the smallest number of iterations for convergence within a specified tolerance. Experience shows that if one plotted the number of iterations of convergence versus  $a$ , the plot would roughly look like a plot of cosecant  $x$  versus  $x$ . Also that  $a_{\min}$  lies in a certain interval and is not unique. For example,  $a_{\min}$  lied in the interval  $[\text{.1329}, \text{.13325}]$  for obtaining the six-iteration convergence of  $X_1$ .

Dolan did not specify his specified tolerance or convergence criteria. The only criteria used here was to take the difference between successive approximations and see if it is less than a certain epsilon, which was  $1E-16$  for our example. Note that it is possible to

converge to a false solution using this difference criteria. One must always check to see if the result satisfies Eq (2.76). An example of this follows from Martensson (Ref 24:31-32).

Example 2.7

$$A = B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad (2.81)$$

Applying the iterative method, we find three results:

$$X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad (2.82)$$

$$X_3 = \begin{bmatrix} 2.77777777777777 & -1.11111111111111 \\ -1.11111111111111 & 1.44444444444444 \end{bmatrix} \quad (2.83)$$

Checking Eq (2.76) we find that  $X_3$  is not a solution. On the other hand, if we apply the  $f(R)$  method, we do find that an  $X_3$  does exist, namely

$$X_3 = 1/9 \begin{bmatrix} 64 & -16 \\ -16 & 13 \end{bmatrix} \quad (2.84)$$

So this procedure is not without flaw. Incorrect solutions can appear.

To explore bounds, on the solutions of the Riccati matrix equation to get an initial starting point, is not of prime interest here. A good treatment of bounds may be found by Kwon and Pearson (Ref 16) and Bellman (Ref 5).

## Section 2. Solutions of Lyapunov Matrix Equations

The Lyapunov matrix equation arises most commonly in the stability analysis of linear and non-linear systems. Other applications include the analysis of beam gridworks (Ref 23). The numerical solution of certain boundary value problems in partial differential equations (Ref 9), and the design of optimal control systems with fixed control structure (Ref 19). The form of the Lyapunov equation we will use is the following:

$$AX - XB = C \quad (2.85)$$

We could easily solve Eq (2.85) by the techniques in the preceding section by setting the matrices associated with the quadratic or higher order terms to zero. Consider the following example with one parameter  $z$ , using the power series method.

### Example 2.8

Let

$$A = \begin{bmatrix} 1 & 1+z \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1+z & 0 \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1-z & -1-z & 1 \\ 0 & 1+z & z \end{bmatrix} \quad (2.86)$$

in Eq (2.85). Then for  $p = 1$

$$X(\lambda, z) = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2.87)$$

$$Y(\lambda, z) = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1-z \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2.88)$$

Hence

$$X = X_0 + X_1 B = \begin{bmatrix} z & z & 1 \\ -1 & 1 & 1-z \end{bmatrix} \quad (2.89)$$

$$Y = Y_0 + AY_1 = \begin{bmatrix} 2 & z & 1 \\ -1 & 1 & 1-z \end{bmatrix} \quad (2.90)$$

The next method gives an explicit solution of Eq (2.85) by using the principal idempotents and nilpotents of the coefficient matrices A and B.

Idempotent-Nilpotent Method. This method was conceived using ideas from Browne (Ref 8:173,184-187,250-251) and relies upon the decomposition of the coefficient matrices into their principal idempotent and nilpotent components. Proof of the method can be found in Jones and Lew (Ref 15).

Suppose A is  $m \times m$ , B is  $n \times n$ , C is  $m \times n$ , and A, B, C belong to the complex vector space of  $p$  by  $q$  matrices. Let A, B have the following representation:

$$A = \sum_{j=1}^{m_1 \leq m} a_j E_j + N_j; \quad B = \sum_{k=1}^{n_1 \leq n} b_k F_k + M_k \quad (2.91)$$

where the  $a_j$  are  $m_1$  non-zero distinct complex eigenvalues and the  $b_k$  are  $n_1$  non-zero distinct complex eigenvalues. The  $\{E_j\}$  and  $\{F_k\}$  form complete sets of orthogonal principal idempotents and the  $\{N_j\}$  and  $\{M_k\}$  form complete sets of orthogonal principal nilpotents. See Browne for method of obtaining idempotents and nilpotents. The following conditions hold for A and B respectively:

$$E_j^2 = E_j; \sum_{j=1}^{m_1 \leq m} E_j = I; E_j E_k = 0, j \neq k$$

$$E_j N_j = N_j = N_j E_j; E_j N_k = N_j N_k = 0, j \neq k \quad (2.92)$$

$$F_k^2 = F_k; \sum_{k=1}^{n_1 \leq n} F_k = I; F_k F_l = 0, k \neq l$$

$$F_k M_k = M_k = M_k F_k; F_k M_l = M_k M_l = 0, k \neq l \quad (2.93)$$

### Theorem 2.8

Let A, B have the representation given in Eqs (2.91), (2.92), (2.93). A sufficient condition for Eq (2.85) to have a solution, is that whenever we have a pair of indices j,k such that  $a_j = b_k$ , then

$$N_j C F_k = E_j C M_k \quad (2.94)$$

$$E_j C F_k = 0 \quad (2.95)$$

$$\sum_{j=1}^{m_1 \leq m} \sum_{k=1}^{n_1 \leq n} \frac{N_j C F_k - E_j C M_k}{a_j - b_k} = 0 \quad (2.96)$$

The solution X is given by

$$X = \sum_{j=1}^{m_1 \leq m} \sum_{k=1}^{n_1 \leq n} \frac{E_j C F_k}{a_j - b_k} \quad (2.97)$$

Example 2.9

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & -2 \\ 4 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (2.98)$$

Then

$$A = 1 \begin{bmatrix} 2/4 & 2/4 & -2/4 \\ -1/4 & 5/4 & -1/4 \\ -3/4 & 3/4 & 1/4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 10/4 & 10/4 & -10/4 \\ 10/4 & 10/4 & -10/4 \end{bmatrix} + 3 \begin{bmatrix} 2/4 & -2/4 & 2/4 \\ 1/4 & -1/4 & 1/4 \\ 3/4 & -3/4 & 3/4 \end{bmatrix}$$

$$= a_1 E_1 + N_1 + a_2 E_2; \quad a_1 = 1; \quad a_2 = 3; \quad N_2 = 0 \quad (2.99)$$

$$B = -2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= b_1 F_1 + b_2 F_2; \quad b_1 = -2; \quad b_2 = -1; \quad M_1 = M_2 = 0 \quad (2.100)$$

Finally

$$X = \frac{E_1 CF_1}{a_1 - b_1} + \frac{E_1 CF_2}{a_1 - b_2} + \frac{E_2 CF_1}{a_2 - b_1} + \frac{E_2 CF_2}{a_2 - b_2}$$

$$= \begin{bmatrix} 0 & 0 \\ 1/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \quad (2.101)$$

is a solution of Eq (2.85).

### III. Reducibility of Matrix Equations Containing Several Parameters with Applications

The purpose of this chapter is to implement the knowledge gained in the preceding chapter to effect a block reduction to any given state matrix containing polynomial elements in several parameters. The two parameter case is dealt with here, and the extension to  $n$  parameters is straightforward. Two methods will be presented. First the Riccati-Lyapunov method, then the Smith Form method. Finally an application to system science will be examined.

#### Section 1. Riccati-Lyapunov Method

This method from Dolan (Ref 11) relies on solutions to the Riccati and Lyapunov equations. Dolan's method is extended to the rectangular case with parameters  $z_1, z_2, \dots, z_q$ .

#### Theorem 3.1

The differential system  $\dot{x} = \tilde{A}x$  in the form

$$\dot{x} = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} x, \quad (3.0)$$

where  $A_{m \times m}$ ,  $B_{n \times n}$ ,  $C_{m \times n}$ ,  $D_{n \times m}$  belong to the ring of matrices  $R[z] = R[z_1, z_2, \dots, z_q]$  is kinematically similar to

$$\dot{y} = \begin{bmatrix} -B-DX & D \\ 0 & XD+A \end{bmatrix} y \quad (3.1)$$

by the transformation

$$x = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} y \quad (3.2)$$

where  $X$  is a solution to the Riccati matrix equation

$$AX + XB + C + XDX = 0 \quad (3.3)$$

Theorem 3.2

The following differential system similar in form to Eq (3.1)

$$\dot{y} = \begin{bmatrix} \tilde{A} & C \\ 0 & \tilde{B} \end{bmatrix} y, \quad (3.4)$$

where  $\tilde{A}_{n \times n}$ ,  $\tilde{B}_{m \times m}$ ,  $C_{n \times m}$  belong to  $R[\underline{z}]$ , is kinematically similar to

$$\dot{z} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix} z \quad (3.5)$$

by the transformation

$$y = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} z \quad (3.6)$$

where  $X$  is a solution to the Lyapunov matrix equation

$$\tilde{A}X - X\tilde{B} = C \quad (3.7)$$

As one can see, the method is a two-stage block reduction. Repeated application of the method to each block and sub-block will finally result in a pure diagonal form.

The following is an example of a complete reduction for the constant case.

### Example 3.1

Reduce

$$\dot{x}(t) = \left[ \begin{array}{cc|c} -3 & -1 & -2 \\ 0 & -2 & 0 \\ \hline 0 & 1 & -1 \end{array} \right] x(t) = \left[ \begin{array}{c|c} -B & D \\ \hline -C & A \end{array} \right] x(t) \quad (3.8)$$

From Example 2.4 a solution to Eq (2.0) is  $X = [0 \ 1]$ . Using Theorem 3.1, Eq (3.8) reduces to

$$\dot{y}(t) = \left[ \begin{array}{cc|c} -3 & -1/3 & -2 \\ 0 & -2 & 0 \\ \hline 0 & 0 & -1 \end{array} \right] y(t) = \left[ \begin{array}{c|c} \tilde{A} & C \\ \hline 0 & \tilde{B} \end{array} \right] y(t) \quad (3.9)$$

Now applying Theorem 3.2, Eq (3.9) reduces to

$$\dot{z}(t) = \left[ \begin{array}{ccc} -3 & -1/3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{array} \right] z(t) = \left[ \begin{array}{c|c} \tilde{A} & 0 \\ \hline 0 & \tilde{B} \end{array} \right] z(t) \quad (3.10)$$

using  $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  as a solution to Eq (3.7).

To finish the reduction we apply Theorem 3.2 to

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & -1/3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \quad (3.11)$$

Using  $X = 1/3$  as a solution to Eq (3.7) we have

$$\dot{w}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} w(t) \quad (3.12)$$

Hence from Eqs (3.9)-(3.12), Eq (3.8) is reduced to

$$\dot{\tilde{z}}(t) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tilde{z}(t) \quad (3.13)$$

which can be easily solved. Next follows a reduction in two parameters  $s$  and  $z$ .

### Example 3.2

Reduce

$$\dot{x}(t) = \left[ \begin{array}{ccc|ccc} s & 0 & 1 & 1 & 1 & 0 \\ 0 & sz+1 & 1 & 1 & 1 & -1 \\ 0 & 0 & z & z-s & z-sz-1 & -1 \\ \hline \text{Circle} & & & s & 0 & 0 \\ & & & 0 & sz+1 & 1 \\ & & & 0 & 0 & z \end{array} \right] x(t) \quad (3.14)$$

Applying Theorem 3.2 with a solution to Eq (3.7), namely

$$X = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad (3.15)$$

we have

$$\dot{y}(t) = \left[ \begin{array}{ccc|ccc} s & 0 & 1 & \bigcirc & & \\ 0 & sz+1 & 1 & & \bigcirc & \\ 0 & 0 & z & & & \bigcirc \\ \hline & & & \bigcirc & s & 0 & 0 \\ & & & & 0 & sz+1 & 1 \\ & & & & 0 & 0 & z \end{array} \right] y(t) \quad (3.16)$$

For now, Eq (3.16) will be left in upper triangular form, even though the system could easily be solved as is using back substitution. Next a reduction method using an equivalence transformation to a Smith form in two parameters  $s$  and  $z$  will be presented. The following treatment, marking a great milestone to reduction, is found in Frost and Storey (Ref 12).

## Section 2. Smith Form Method

The results here involve transformations of equivalence between  $pxq$  matrices  $A(s,z)$  and  $B(s,z)$  of the form

$$B(s,z) = M(s,z)A(s,z)N(s,z) \quad (3.17)$$

where  $M(s,z)$  and  $N(s,z)$  are unimodular matrices. The equivalence transformation of most interest is where  $B(s,z)$  is a type of Smith form of  $A(s,z)$  over  $R[s,z]$ . The Smith Form method for two parameters is much like the method for the one-parameter case but requires no zeros of a matrix over  $R[s,z]$  for equivalence.

### Theorem 3.3

A  $pxq$  matrix  $A(s,z)$  of rank  $t$  over  $R[s,z]$  can be reduced by elementary transformations to the Smith form

$$S(s,z) = \begin{cases} \begin{bmatrix} E(s,z) & 0 \end{bmatrix} , & p < q \\ E(s,z) , & p = q \\ \begin{bmatrix} E(s,z) \\ 0 \end{bmatrix} , & p > q \end{cases} \quad (3.18)$$

where

$$E(s,z) = \text{diag}[e_i(s,z)] \quad (3.19)$$

The diagonal elements of  $E(s,z)$  are the invariant polynomials over  $R[s,z]$  of  $A(s,z)$ , given by

$$e_i(s,z) = \frac{d_i(s,z)}{d_{i-1}(s,z)} , \quad i = 1, 2, \dots, t \quad (3.20)$$

where  $d_0 \equiv 1$  and the determinantal divisor  $d_i$  ( $i = 1, 2, \dots, t$ ) is the greatest common denominator of all the  $i$ th-order minors of  $A(s,z)$ , and where  $e_i(s,z)$  is a divisor of  $e_{i+1}(s,z)$  for all  $i < t$ . That is, there exists unimodular matrices  $M(s,z)$  and  $N(s,z)$  over  $R[s,z]$  such that

$$M(s,z)A(s,z)N(s,z) = S(s,z) \quad (3.21)$$

#### Theorem 3.4

Two matrices  $A(s,z)$  and  $B(s,z)$  are equivalent over  $R[s,z]$  if there exists unimodular matrices  $M(s,z)$  and  $N(s,z)$  such that Eq (3.17) is true.

It is easy to show that matrices are equivalent over  $R[s]$  if and only if they have the same Smith form, but it does not mean that matrices over  $R[s,z]$  that have the same Smith form are equivalent. See the following example.

Example 3.3

$$A(s,z) = \begin{bmatrix} s & 0 & 1 \\ 0 & sz+1 & 1 \\ 0 & 0 & z \end{bmatrix} \quad B(s,z) = \begin{bmatrix} s & 0 & 0 \\ 0 & sz+1 & 1 \\ 0 & 0 & z \end{bmatrix} \quad (3.22)$$

The Smith form for  $A(s,z)$  and  $B(s,z)$  is given by

$$S(s,z) = S_A(s,z) = S_B(s,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & sz(sz+1) \end{bmatrix} \quad (3.23)$$

$B(s,z)$  and  $A(s,z)$  have the same Smith form, but there does not exist  $M(s,z)$  and  $N(s,z)$  such that Eq (3.17) holds. Referring back to Example 3.2, to complete the reduction via Smith form, replace block forms by form in Eq (3.23). To get a stronger condition on equivalence via the Smith form, the definition of zeros of a matrix are needed.

Zeros of a Matrix. On removal of the determinantal divisor  $d_i(s,z)$  from all the  $i$ th-order minors of a matrix  $A(s,z)$ , the remaining polynomials may all be simultaneously zero for one or more values of the pair  $(s,z)$ . Such a value of  $(s,z)$  will be denoted an  $i$ th-order zero of  $A(s,z)$ .

Example 3.4

$$A(s,z) = \begin{bmatrix} s+z & 0 & z \\ 0 & s+z & 0 \\ 0 & 0 & s \end{bmatrix} \text{ has the same Smith form as the matrix}$$

$$B(s,z) = \begin{bmatrix} s+z & 0 & 1 \\ 0 & s+z & 0 \\ 0 & 0 & s \end{bmatrix}, \text{ but } A(s,z) \text{ has first and second-order zero}$$

$(0,0)$ , whereas  $B(s,z)$  has no first-order zeros and many second-order zeros.

It is important to see that matrices over  $R[s,z]$  which have the same Smith form over  $R[s,z]$  but do not have the same zeros are not equivalent over  $R[s,z]$ . The equivalence transformation over  $R[s,z]$  preserves the zeros of a matrix over  $R[s,z]$ . This brings us to the following theorem.

### Theorem 3.5

Two matrices  $A(s,z)$  and  $B(s,z)$  are equivalent over  $R[s,z]$  iff  $A(s,z)$  and  $B(s,z)$  have no zeros and have the same Smith form.

The following is an example of reduction to Smith form using Theorem 3.3.

### Example 3.5

Reduce

$$\dot{x}(t) = \begin{bmatrix} s+1 & 1+z(s+1) & 0 \\ s & sz+1 & -(s+1)(s+z) \\ 0 & s(s+1) & s+1 \end{bmatrix} x(t) \quad (3.24)$$

using

$$M(s,z) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ s^2(s+1) & -s(s+1)^2 & 1 \end{bmatrix} \quad (3.25)$$

$$N(s,z) = \begin{bmatrix} -z & -[1+z(s+1)] & -(s+1)(s+z)[1+z(s+1)] \\ 1 & s+1 & (s+1)^2(s+z) \\ 0 & 0 & 1 \end{bmatrix}$$

We have, using Theorem 3.3,

$$\dot{y}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+1+s(s+1)^3(s+z) \end{bmatrix} y(t) \quad (3.26)$$

The Smith form works well with matrices containing parameters but one has to work hard to get the M and N matrices. Using Theorem 3.3 as a basis, one can easily extend the Smith form to the n-parameter case by induction. This is shown by the following theorems.

Theorem 3.3A

A  $p \times q$  matrix  $A(\underline{z})$  of rank  $t$  over  $R[\underline{z}]$  can be reduced by elementary transformations to the Smith form

$$S(\underline{z}) = \begin{cases} \begin{bmatrix} E(\underline{z}) & 0 \end{bmatrix} & , p < q \\ E(\underline{z}) & , p = q \\ \begin{bmatrix} E(\underline{z}) \\ 0 \end{bmatrix} & , p > q \end{cases} \quad (3.18a)$$

where

$$E(\underline{z}) = \text{diag}[e_i(\underline{z})] \quad (3.19a)$$

The diagonal elements of  $E(\underline{z})$  are the invariant polynomials over  $R[\underline{z}]$  of  $A(\underline{z})$ , given by

$$e_i(\underline{z}) = \frac{d_i(\underline{z})}{d_{i-1}(\underline{z})}, \quad i = 1, 2, \dots, t \quad (3.20a)$$

where  $d_0 \equiv 1$  and the determinantal divisor  $d_i$  ( $i = 1, 2, \dots, t$ ) is the

greatest common denominator of all the  $i$ th-order minors of  $A(\underline{z})$ , and where  $e_i(\underline{z})$  divides  $e_{i+1}(\underline{z})$  for all  $i < t$ . That is there exists unimodular matrices  $M(\underline{z})$  and  $N(\underline{z})$  over  $R[\underline{z}]$  such that

$$M(\underline{z})A(\underline{z})N(\underline{z}) = S(\underline{z}) \quad (3.21a)$$

#### Theorem 3.5A

Two matrices  $A(\underline{z})$  and  $B(\underline{z})$  are equivalent over  $R[\underline{z}]$  iff  $A(\underline{z})$  and  $B(\underline{z})$  have no zeros and have the same Smith form.

### Section 3. Application of Reducibility to Singular Systems

In this section we will illustrate the applicability of previous theorems by use of an example in systems science. The following treatment extends Lovass-Nagy (Ref 21).

Consider the linear system

$$D(\underline{z})\dot{x}(t) = A(\underline{z})x(t) + B(\underline{z})u(t) \quad (3.27)$$

where  $x_{n \times 1}$ ,  $A_{m \times n}$ ,  $B_{m \times p}$ ,  $D_{m \times n}$  and  $u_{p \times 1}$  belong to  $R[z_1, z_2, \dots, z_q]$ , or in shorthand,  $R[\underline{z}]$ . Equation (3.27) could be considered as the state equation of a control problem, where one hopes to find an input  $u(t)$  that will force  $x(t)$  to some prescribed function of  $t$ . The following theorems are necessary to analyze Eq (3.27).

#### Theorem 3.6

The equation

$$AXF = B \quad (3.28)$$

has a solution if and only if

$$AA^{-1}BF^{-1} = B \quad (3.29)$$

and the general solution is given by

$$X = A^{-1}BF^{-1} + W - A^{-1}AWFF^{-1} \quad (3.30)$$

where W is an arbitrary  $n \times 1$  matrix.

### Theorem 3.7

The matrix equation

$$Ax = b \quad (3.31)$$

has a solution if and only if

$$(I - AA^{-1})b = 0 \quad (3.32)$$

and the general solution is given by

$$X = A^{-1}b + (I - A^{-1}A)h \quad (3.33)$$

where h is an arbitrary  $n \times 1$  matrix.

An example of the reduction of Eq (3.27) follows.

### Example 3.6

Let  $A(\underline{z})$ ,  $B(\underline{z})$ ,  $D(\underline{z})$  over  $R[\underline{z}]$  be as follows:

$$A = \begin{bmatrix} a_1(\underline{z}) & c(\underline{z}) \\ 0 & a_2(\underline{z}) \end{bmatrix}, B = \begin{bmatrix} 1 \\ b(\underline{z}) \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad (3.34)$$

where  $a_1$ ,  $a_2$ ,  $c$  and  $b$  are polynomial functions in the parameters  $z_1, \dots, z_q$  with coefficients in the field of real numbers. The system in Eq (3.27) can be reduced to an equivalent system by the

transformation  $x = Py$ , where

$$P = \begin{bmatrix} 1 & -X \\ 0 & 1 \end{bmatrix} \quad (3.35)$$

and  $X$  is a solution to the Lyapunov equation

$$a_1(\underline{z})X - Xa_2(\underline{z}) = c(\underline{z}) \quad (3.36)$$

Using the transformation we have

$$DP\dot{y} = APy + Bu \quad (3.37)$$

Premultiplying by  $P^{-1}$  gives

$$P^{-1}DP\dot{y} = P^{-1}APy + P^{-1}Bu \quad (3.38)$$

Expanding we have

$$\begin{bmatrix} 1+X & -X-X^2 \\ 1 & -X \end{bmatrix} \dot{y} = \begin{bmatrix} a_1(\underline{z}) & 0 \\ 0 & a_2(\underline{z}) \end{bmatrix} y + \begin{bmatrix} 1+Xb(\underline{z}) \\ b(\underline{z}) \end{bmatrix} u \quad (3.39)$$

$$\tilde{D}\dot{y} = \tilde{A}y + \tilde{B}u \quad (3.40)$$

Using this decoupled system the procedure to find  $u(t)$  is much simplified. For simplification let  $X = 0$ . It is easy to see that a generalized inverse of  $\tilde{B}$  is

$$\tilde{B}^- = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (3.41)$$

and that

$$(I - \tilde{B}\tilde{B}^-) = \begin{bmatrix} 0 & 0 \\ -b(\underline{z}) & 1 \end{bmatrix} \quad (3.42)$$

The consistency condition of Eq (3.31) implies that Eq (3.39) has a solution  $u$  if and only if

$$(I - \tilde{B}\tilde{B}^T)(\tilde{D}\dot{\tilde{y}} - \tilde{A}\tilde{y}) = 0 \quad (3.43)$$

or expanding

$$\begin{bmatrix} 0 & 0 \\ -b(\underline{z}) & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} - \begin{bmatrix} a_1(\underline{z}) & 0 \\ 0 & a_2(\underline{z}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = 0 \quad (3.44)$$

If  $y_1$  and  $y_2$  are any two functions satisfying Eq (3.43), then a particular solution of  $u(t)$  is given by

$$u(t) = \tilde{B}^T(\tilde{D}\dot{\tilde{y}} - \tilde{A}\tilde{y}) \quad (3.45)$$

or expanding

$$u(t) = [1 \ 0] \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} - \begin{bmatrix} a_1(\underline{z}) & 0 \\ 0 & a_2(\underline{z}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) \quad (3.46)$$

$$u(t) = [1 \ 0] \begin{bmatrix} \dot{y}_1 - a_1(\underline{z})y_1 \\ \dot{y}_2 - a_2(\underline{z})y_2 \end{bmatrix} = \dot{y}_1 - a_1(\underline{z})y_1 \quad (3.47)$$

#### IV. Conclusions and Recommendations

Reducibility of matrix equations containing several parameters have been studied. Two reduction methods used were the Riccati-Lyapunov method and the Smith Form method. The Riccati-Lyapunov method used iterative and non-iterative schemes to find solutions to the Riccati and Lyapunov equations. This method to effect a reduction works quite well. Follow-on work should be the formulation and comparison of computer algorithms that demonstrate the non-iterative solution schemes delineated in Chapter II. For the iterative scheme, the regions of convergence of parameter  $a$  should be examined more closely, the method should be extended to include cubic and higher order terms, and ways of obtaining bounds and initial starting matrices should be explored. The Smith Form method used an equivalence transformation and is an alternate form of reduction that can do the reduction in one step as opposed to two in the previous method. One cannot say that the Smith Form method is easier than the other because they are equally difficult. As it is a lengthy process to obtain solutions of the Riccati and Lyapunov equations, it is also laborious to find matrices  $M$  and  $N$  to effect a reduction. Furthermore the effort is multiplied when the matrices contain multiple parameters. Nevertheless the Smith Form method works very well. Future work concerning the Smith Form method should include the formulation of a computer algorithm that will reduce a given  $pxq$  matrix containing several parameters to Smith form.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM									
1. REPORT NUMBER AFIT/GE/MA/81D-1	2. GOVT ACCESSION NO. <b>A115 568</b>	3. RECIPIENT'S CATALOG NUMBER									
4. TITLE (and Subtitle) REDUCIBILITY OF MATRIX EQUATIONS CONTAINING SEVERAL PARAMETERS		5. TYPE OF REPORT & PERIOD COVERED MS Thesis									
		6. PERFORMING ORG. REPORT NUMBER									
7. AUTHOR(s) CHARLES A. LEW Lieutenant, USAF		8. CONTRACT OR GRANT NUMBER(s)									
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology (AFIT/EN) Wright-Patterson AFB OH 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS									
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE December 1981									
		13. NUMBER OF PAGES									
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified									
		15a. DECLASSIFICATION/ DOWNGRADING SCHEDULE									
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.											
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) <div style="text-align: right;">Dean for Research and Professional Development Air Force Institute of Technology (ATC) Wright-Patterson AFB, OH 45433</div> <div style="text-align: center;"><b>15 APR 1982</b> <i>Sp. E. Wolan</i></div>											
18. SUPPLEMENTARY NOTES Approved for public release: IAW AFR 190-17 <div style="text-align: right;"><del>F. G. LYNCH, Major, USAF</del> <del>Director of Information</del></div>											
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) <table border="0" style="width: 100%;"> <tr> <td>Reducibility</td> <td>Lyapunov Equations</td> <td>Generalized Equations</td> </tr> <tr> <td>Matrix Equations</td> <td>Smith Form</td> <td>Block Diagonalization</td> </tr> <tr> <td>Riccati Equations</td> <td>Polynomial Rings</td> <td>High Order Matrix Equations</td> </tr> </table>			Reducibility	Lyapunov Equations	Generalized Equations	Matrix Equations	Smith Form	Block Diagonalization	Riccati Equations	Polynomial Rings	High Order Matrix Equations
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